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Differential Equation for Continuous Normalization

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When the response of a dynamical system is seemingly random and is confined in a finite region and shows extreme sensitivity to small changes in initial conditions, we say that the motion is chaotic. To represent chaos of a system precisely and quantitatively, we employ measurement quantity that represents the system's degree of chaos such as a Lyapunov exponent. A way of computing a Lyapunov exponent employs periodic renormalization for the state perturbation vector. However, the application of periodic renormalization for a Lyapunov exponent computation poses difficulties. One difficulty is exponential growth of the norm of the state perturbation vector. A common approach for avoiding this computational problem is periodic renormalization. However, periodic renormalization raises a discontinuity in the state perturbation vector that is not a standard case in optimal control theory as one wants to extremize chaos by manipulating a Lyapunov exponent. To circumvent the exponential growth in magnitude and the state perturbation vector discontinuity problem, one may employ a method of "continuous normalization" which replaces periodic discontinuous renormalization with differential equations that correspond to continuous normalization at each instant of time. This study provides details concerning the development of continuous normalization technique and presents an example for some systems. Also the comparison between the result produced by continuous normalization and that by the periodic renormalization of the state perturbation vector will be given.

Key Words: Chaos, Lyapunov Exponent, Periodic Renormalization, Linear Perturbation, Continuous Normalization, Optimal Control

1. Lyapunov Exponents and Linear Perturbations

The Lyapunov numbers or exponents are time-averaged quantities characterizing the behavior of an attractor. The Lyapunov numbers generalize the idea of eigenvalues or singular values (Greene, 1987) of linear systems to the more complex case of nonlinear systems. The basic definition of a Lyapunov exponent in discrete-time systems is presented in (Schuster, 1989).

Consider a scalar mapping of the form

$$x_{k+1}=f(x_k). \tag{1}$$

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Generally, any numerical integrator for differential equations is considered as a discrete-time mapping of the form (1) in the sense that the precision is finite and the integration is calculated in a certain interval of time. The Lyapunov number is a time average quantity which measures the separation of two adjacent points produced by the mapping, as shown in Fig. 1.

The separation of initially adjacent points grows exponentially by the factor of a certain number. After N iterations, the separation is obtained as

$$\varepsilon e^{N\sigma(x_0)}=|f^N(x_0+\varepsilon)-f^N(x_0)|, \tag{2}$$

where ε is an arbitrarily small number. The Lyapunov number involves a double limiting process for the rate of exponential separation of two initially adjacent points as $N\to\infty$ and $\varepsilon\to$

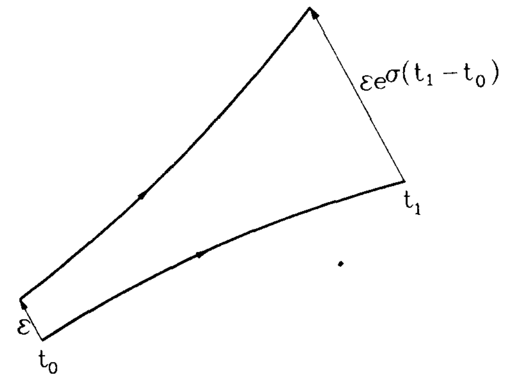

Fig. 1 Two initially adjacent points become separated exponentially

0 and is expressed as

$$\sigma(x_0)=\lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{N}\ln\left|\frac{f^N(x_0+\varepsilon)-f^N(x_0)}{\varepsilon}\right|$$
$$=\lim_{N\to\infty}\frac{1}{N}\ln\left|\frac{df^N(x_0)}{dx_0}\right|, \quad (3)$$

if $f(\cdot)$ is differentiable. Using the chain rule one can get

$$\frac{d}{dx}f^N(x)\Big|_{x_0}=\frac{d}{dx}f[f\cdots f(x)]$$
$$=f'(x_{N-1})f'(x_{N-2})\cdots f'(x_0) \quad (4)$$

where $f'(x)=df(x)/dx$. Thus

$$\sigma(x_0)=\lim_{N\to\infty}\frac{1}{N}\ln\left|\prod_{i=0}^{N-1}f'(x_i)\right|$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\ln|f'(x_i)| \quad (5)$$

For a system of dimension $n>1$, by expanding the result in (5), the spectrum of the Lyaapunov numbers is defined as (Schuster, 1989) :

$$(e^{\sigma_1},e^{\sigma_2},\cdots,e^{\sigma_n})=\lim_{N\to\infty}[(\text{magnitude of eigenvalues of } \prod_{j=0}^{N-1}J(x_j))^{1/N}], \quad (6)$$

where $J(x)=\partial f(x)/\partial x$.

To discuss Lyapunov exponents for a continuous dynamical system, consider a system of the form

$$\dot{x}=f(x) \quad (7)$$

assuming that $f(x)$ is continuous and differentiable for all x. The Lyapunov exponents of the continuous-time dynamical system (7) are defined by letting $\delta(t)$ denote the distance between two trajectories for a continuous-time dynamical system. If there exists a number σ which satisfies

$$\delta(t)\to\delta(0)e^{\sigma t} \;\; as \;\; t\to\infty \quad (8)$$

for arbitrary small $\delta(0)$, then σ is called a Lyapunov exponent.

For an n-dimensional dynamical system, $n>1$, there are n Lyapunov exponents, corresponding to an n-dimensional orthogonal reference frame. To define Lyapunov exponents, one needs to consider an n-dimensional ellipsoid centered at a point $x(t)$, the solution of Eq. (7) along a trajectory. By a similar way as used for one-dimensional systems, let $\delta_i(t)$ denote the length of the i^{th} semi-axes of the ellipsoid with $\delta_i(0)=\delta_0$, $i=1,\cdots,n$. The spectrum of Lyapunov exponents is defined by

$$\sigma_i=\lim_{t\to\infty}\left(\lim_{\delta_i(0)\to 0}\frac{1}{t}\ln\left(\frac{\delta_i(t)}{\delta_i(0)}\right)\right),\; i=1,\cdots,n, \quad (9)$$

if the double limits exist, where the indices are ordered so that $\sigma_1\leqq\sigma_2\leqq\cdots\leqq\sigma_n$. Note that the definition in (9) is similar to that in (3) except for the fact that N in (3) is replaced by t. If Δt is the time required for one iteration of (3), then the time required for N iteration is $N\Delta t$.

2. Gramm-Schmidt Reorthonomalization for Lyapunov Exponents

Consider a system of the form (7) and let $x(t)=\psi(t, x(0))$ be a solution. To compute the Lyapunov exponents, consider an n-dimensional ellipsoid whose center lies on the reference trajectory $x(t)$ and semi-axes are determined by n-orthogonal perturbation vectors. Let $\underline{x}(t)=\psi(t, \underline{x}(0))$ be the solution of (7) generated with the initial value $\underline{x}(0)$ being arbitrarily close to $x(0)$.

Applying Taylor's theorem, assuming the right hand side of (7) is continuous and continuously differentiable along $x(t)$, we have

$$\underline{x}(t)=x(t)+\varepsilon\eta(t)+O(\varepsilon), \quad (10)$$

where $O(\varepsilon)/\varepsilon\to 0$ and $\eta(t)\in E^n$ satisfies the linearized state perturbation equations

$$\dot{\eta}=D(t)\eta$$
$$D(t)=\frac{\partial f(x)}{\partial x} \quad (11)$$

along $\boldsymbol{x}(t)$. Replacing $\delta_i(0)$ with $\underline{\boldsymbol{x}}(0)-\boldsymbol{x}(0)$ and $\delta_i(t)$ with $\underline{\boldsymbol{x}}(t)-\boldsymbol{x}(t)$ in Eq. (9), we have

$$\sigma=\lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{\|\boldsymbol{\eta}(t)\|}{\|\boldsymbol{\eta}(0)\|}\right) \quad (12)$$

as $\varepsilon\to 0$. Note that Eq. (12) handles the limiting process for $\delta_i(0)\to 0$ in (9) automatically.

By the definition of the Lyapunov exponents, semi-axes grow exponentially with time and diverge in magnitude, if the Lyapunov exponent corresponding to that semi-axis is a positive number, beyond the capacity that a finite-word-length computer can handle. This is not a mathematical problem but a computational problem (Wolf, 1985). Another peculiarity (both mathematical and computational) is that each perturbation vector has a tendency, over time, to align itself along the direction corresponding to the largest Lyapunov exponent. The first problem may be circumvented by renormalization of the perturbation vectors when their magnitudes become big. The second problem can be solved by repeated use of reorthogonalization on the perturbation vectors. A method of Gramm-Schmidt reorthonormalization is presented in the next paragraph.

To compute the spectrum of Lyapunov exponents, using the method in (Wolf, 1985), we choose n orthogonal unit vectors $\boldsymbol{\eta}_i(t)$, $i=1,\cdots,n$, with $\varepsilon\boldsymbol{\eta}_i(0)$ representing the n semi-axes of an arbitrary small initial (spherical) ellipsoid. Then the state perturbation Eq.(11) is used to generate corresponding perturbation solutions $\boldsymbol{\eta}_i(t)$. To avoid numerical overflow, the integrations in (11) are restarted at convenient times $t=k\Delta T$, after the results have been orthogonalized, $\boldsymbol{\eta}_i(t)\to\boldsymbol{\eta}_i'(t)$, and renormalized, $\boldsymbol{\eta}_i'(t)/\|\boldsymbol{\eta}_i'(t)\|\to\boldsymbol{\eta}_i(t)$, using a Gramm-Schmidt reorthonomalization process :

$$\begin{aligned}
&\boldsymbol{\eta}_1'=\boldsymbol{\eta}_1\\
&\rho_1=\|\boldsymbol{\eta}_1'\|\\
&\boldsymbol{\eta}_1=\boldsymbol{\eta}_1'(t)/\|\boldsymbol{\eta}_1'\|\\
&\quad\vdots\\
&\boldsymbol{\eta}_l'=\boldsymbol{\eta}_l-\sum_{j=1}^{l-1}(\boldsymbol{\eta}_l^T\boldsymbol{\eta}_j)\boldsymbol{\eta}_j,\ l=2,\cdots,n\\
&\rho_l=\|\boldsymbol{\eta}_l'\|\\
&\boldsymbol{\eta}_l=\boldsymbol{\eta}_l'(t)/\|\boldsymbol{\eta}_l'(t)\|,
\end{aligned} \quad (13)$$

where all quantities are evaluated at time $t_k=k\Delta T$, $k=1, 2, \cdots$. The lengths $\rho_i(t_k)$ of the orthogonalized vectors, prior to rescaling, are the lengths of the transformed ellipsoid semi-axes after a time interval ΔT, starting from a unit sphere at time t_{k-1}. From (9) and (13) the Lyapunov exponents can be calculated as

$$\sigma_i=\lim_{K\to\infty}\left(\frac{1}{K\Delta T}\sum_{k=1}^{K}\ln(\rho_i(k\Delta T))\right). \quad (14)$$

3. Differential Equation for the Largest Lyapunov Exponent

Let $\gamma_i(t)$ and $\boldsymbol{v}_i(t)$ be the eigenvalue and unit eigenvector of the state transition matrix, $\boldsymbol{\Phi}(t, 0)$, which is the solution of the $n\times n$ system of differential equations

$$\frac{\partial\boldsymbol{\Phi}(t,0)}{\partial t}=\boldsymbol{D}(t)\boldsymbol{\Phi}(t,0),\quad \boldsymbol{\Phi}(0,0)=\boldsymbol{I}. \quad (15)$$

Then $\boldsymbol{\eta}_i(t)$, a solution to (11) along $\boldsymbol{x}(t)=\boldsymbol{\psi}(t, \boldsymbol{x}(0))$, is given by

$$\boldsymbol{\eta}_i(t)=\boldsymbol{\Phi}(t,0)\boldsymbol{\eta}_i(0). \quad (16)$$

Choose $\boldsymbol{\eta}_i(0)=\boldsymbol{v}_i(t)$ in (16) yielding

$$\boldsymbol{\eta}_i(t)=\boldsymbol{\Phi}(t,0)\boldsymbol{\eta}_i(0)=\gamma_i(t)\boldsymbol{\eta}_i(0). \quad (17)$$

From (17) and the definition of Lyapunov exponents in (12) we have

$$|\gamma_i(t)|\to e^{\sigma_i t}\ as\ t\to\infty.$$

Now consider an arbitrary initial vector that can be expressed, as a linear combination of the eigenvectors of $\boldsymbol{\Phi}$(t,0), as

$$\boldsymbol{\eta}(0)=\sum_{i=1}^{n}\alpha_i\boldsymbol{v}_i(t). \quad (18)$$

At time t, we have

$$\boldsymbol{\eta}(t)=\boldsymbol{\Phi}(t,0)\boldsymbol{\eta}(0)=\alpha_1\gamma_1(t)\boldsymbol{v}_1(t)+\cdots+\alpha_n\gamma_n(t)\ \boldsymbol{v}_n(t).$$

Since $\|\alpha_i\gamma_i(t)\boldsymbol{v}_i(t)\|\to e^{\sigma_i(t)}\|\alpha_i\boldsymbol{v}_1(t)\|$, $i=1,\cdots,n$, and $e^{\sigma_1 t}$ dominates (recall that $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_n$), we have

$$\|\boldsymbol{\eta}_1(t)\|\to e^{\sigma_1 t}\|\alpha_1\ \boldsymbol{v}_1(t)\|\text{ as }t\to\infty. \quad (19)$$

Due to this fact, we can choose an initial perturbation vector arbitrarily if we want to compute only the largest Lyapunov exponent.

To obtain a differential equation for the largest Lyapunov exponent, we differentiate σ in (12) with respect to t:

$$\sigma=\frac{d}{dt}\left[\frac{1}{t}\ln\frac{\|\eta\|}{\|\eta_0\|}\right]$$
$$=\frac{1}{t}\frac{\eta^T\dot{\eta}}{\|\eta\|^2}-\frac{1}{t^2}\ln\frac{\|\eta\|}{\|\eta_0\|}$$
$$=\frac{1}{t}\frac{\eta^T\dot{\eta}}{\|\eta\|^2}-\frac{\sigma}{t} \tag{20}$$

provided that $t\neq0$.

4. Continuous Normalization for the Largest Lyapunov Exponent

As discussed earlier, state perturbations grow exponentially. To remedy this problem, one may renormalize state perturbation vectors to unit vectors periodically (Wolf, 1985). However, this renormalization procedure for the computation of Lyapunov exponents causes discontinuities in the state variables that is not a standard case in optimal control theory (Pontryagin, et. al., 1964) as one wants to extremize chaos by mainipulating a Lyapunov exponent. In this paper, we present a method of "continuous normalization" which replaces periodic discontinuous renormalization with differential equations that correspond to continuous renormalization at each time instance (Lee, 1991).

Let $\xi=\eta/\|\eta\|$ be a normalized perturbation vector. Differentiating ξ with respect to time, we get

$$\dot{\xi}=\frac{\dot{\eta}}{\|\eta\|}-\frac{\eta\eta^T\dot{\eta}}{\|\eta\|^3}$$
$$=\left(I-\frac{\eta\eta^T}{\|\eta\|^2}\right)\frac{\dot{\eta}}{\|\eta\|} \tag{21}$$

Substituting $\dot{\eta}$ from (11) into (21), we obtain

$$\dot{\xi}=(I-\xi\xi^T)D(t)\xi \tag{22}$$

and corresponding $\dot{\sigma}$ in (20)

$$\dot{\sigma}=\frac{1}{t}\xi^TD(t)\xi-\frac{\sigma}{t} \tag{23}$$

provided that $t\neq0$. Note that ξ only contains directional information for η. One may include a differential equation for the magnitude of η, i.e. $d(\|\eta\|)/dt$, if it is needed. Advantages of this approach are the magnitude of the perturbation vector ξ stays unity and there is no discontinuity in the state perturbation vector.

It is still undesirable to have t in the denominator of Eq. (23) which prevents choosing $t_0=0$. To relieve this difficulty, we integrate (23), using the method of integration factors. Moving the term σ/t in (23) to the left and multiplying both sides by t, we have

$$t\left(\dot{\sigma}+\frac{\sigma}{t}\right)=\frac{d}{dt}(t\sigma)=\xi^TD(t)\xi \tag{24}$$

With an arbitrary choice of t_0 and $t_f\neq0$, we finally have

$$\sigma(t_f)=\frac{1}{t_f}\int_{t_0}^{t_f}\xi^TD(t)\xi dt+\frac{\sigma(t_0)t_0}{t_f}. \tag{25}$$

Note that we may choose $t_0=0$ or $\sigma(t_0)=0$ to nullify the right most term of (25). However, it will vanish as $t_f\to\infty$ so that we can ignore it for Lyapunov exponent computation purposes. Final form of the differential equation for the largest Lyapunov exponent is

$$\dot{\sigma}=\xi^TD(t)\xi. \tag{26}$$

Note that Eq. (26) does not include $1/t_f$ as in Eq. (25). When the largest Lyapunov exponent is required after the integration is finished, we divide the resulting σ in Eq. (26) by t_f.

5. Numerical Study

To verify the method developed here, we compute the largest Lyapunov exponent for well-known systems and compare them with those computed by the method of periodic normalization. Many researchers have studied Duffing's oscillator (Dowell, 1989, Holms, 1983, Moon, 1985, Pezeshki, 1988) since it not only has very rich chaotic characteristics but also represents real physical systems. Duffing's equation chosen is

$$\ddot{y}+\delta\dot{y}-\frac{1}{2}(y-y^3)=P\sin\omega t \tag{27}$$

or, in a state space form,

$$\dot{x}_1=x_2$$
$$\dot{x}_2=\frac{1}{2}(x_1-x_1^3)-\delta x_2+P\sin\omega t, \tag{28}$$

a system of differential equation for the normalized state perturbations

$$\dot{\xi} = (I - \xi\xi^T)D(t)\xi = \begin{bmatrix} \xi_2 - \xi_1^2\xi_2 - {}^1/_2(1-3x_1^2)\xi_1^2\xi_2 + \delta\xi_1\xi_2^2 \\ -\xi_1\xi_2^2 + {}^1/_2(1-3x_1^2)(1-\xi_2^2)\xi_1 - \delta\xi_2(1-\xi_2^2) \end{bmatrix}, \tag{29}$$

a differential equation for the largest Lyapunov exponent

$$\dot{\sigma} = \xi^T D(t)\xi = \frac{1}{2}(1-3x_1^2)\xi_1\xi_2 + \xi_1\xi_2 - \delta\xi_2^2, \tag{30}$$

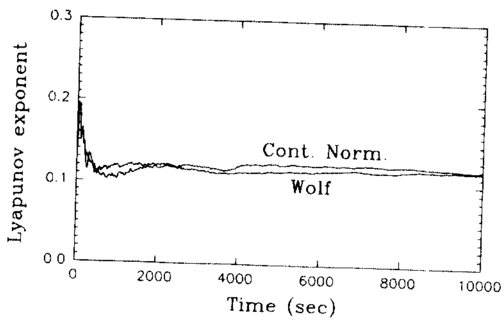

Fig. 2 The first Lyapunov exponent of Duffing's system computed by Wolf's method and Continuous Normalization technique

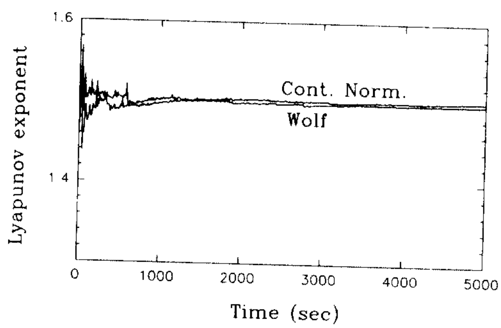

Fig. 3 The first Lyapunov exponent of Lorenz's system computed by Wolf's method and Continuous Normalization technique

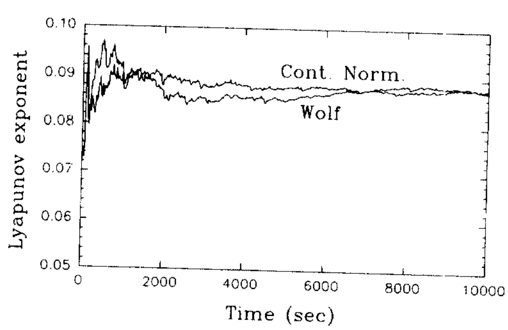

Fig. 4 The first Lyapunov exponent of Rössler's system computed by Wolf's method and Continuous Normalization technique

and parameter values $\delta=0.168$, $P=0.25$, and $\omega=1$.

Adams' variable-order, variable-step integration method (Shampine,1975) is employed to integrate Eqs. (28), (29), and (30), with the local error controlled to less than 1×10^{-9}, from $t=0$ to $t_f=10{,}000s$ with initial values $x_1(0)=x_2(0)=0$, $\xi_1(0)=1$, $\xi_2(0)=0$. We also compute the largest Lyapunov exponent for known systems such as Lorenz's system (Lorenz, 1963, Gibbon, 1980, Curry, 1978)

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= Rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{31}$$

with

$a=16$, $R=45.92$, $b=4$, $(x_1,x_2,x_3)_{t=0}=(10, 10, 20)$

and Rössler's system (Schuster, 1989)

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 - ax_2 \\ \dot{x}_3 &= b + x_1x_3 - cx_3 \end{aligned} \tag{32}$$

with

$a=0.15$, $b=0.2$, $c=10.0$, $(x_1,x_2,x_3)_{t=0}=(0, 0,0)$

to verify the continuous normalization technique. Figures of the first Lyapunov exponents calculated by the method presented in (Wolf, 1985) and by the continuous normalization technique which is formulated in (25) or (26) are presented in Figs. 2, 3, and 4, respectively.

6. Conclusion and Discussion

The current reorthonormalization procedure for the calculation of Lyapunov exponents induces discontinuous jump changes in the variables of the state perturbations. To circumvent state perturbation discontinuities, we developed a continuous normalization technique which replaces periodic discontinuous renormalization with differential equations that correspond to continuous normalization. To verify the method developed here, we computed the largest Lyapunov exponent for well-known systems such as Duffing's oscillator, Lorenz's system, and

Rössler's system then compare the values with those calculated by the method developed in (Wolf, 1985). We could see that two methods well agree for the systems tested.

Justification for the method developed in this paper lies in that, in order to apply current optimal control theory, the quantity which is to be extremized not only should be in a proper form, namely in an integral form but also should be continuous along the trajectory of the solution. The differential equation for the largest Lyapunov exponent developed in the paper can be used as a performance index in an optimal control designed either to make the system more chaotic or to eliminate the unwanted fluctuating chaotic motion.

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